Statistics for the stochastic wave equation

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\lim_{\alpha\to\infty} \frac{1}{2\pi} \int_{0}^{2\pi} f(x) \beta(\alpha x) \mathrm{d}x &= \left(\tfrac{1}{2\pi} \int_0^{2\pi} f(x) \mathrm{d}x\right) \left(\tfrac{1}{2\pi} \int_0^{2\pi} \beta(x) \mathrm{d}x\right)
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\lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{0}^{2\pi} f(x)\beta(\alpha x)\mathrm{d}x &= \left(\tfrac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x\right)\left(\tfrac{1}{2\pi}\int_0^{2\pi} \beta(x)\mathrm{d}x\right)\\
\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{nx}{2}\right)}{n\sin^2\left(\frac{x}{2}\right)}f(x)\mathrm{d}x
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\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{nx}{2}\right)}{n\sin^2\left(\frac{x}{2}\right)}f(x)\mathrm{d}x &= 2\pi\sum_{k\in\mathbb{Z}} f(2\pi k)
\end{aligned}
$$

Riemann–Lebesgue on the Torus

Lemma (Riemann–Lebesgue on $\mathbb{T}$)

For $f \in L^1(\mathbb{T})$, we have $|\mathcal{F}(f)(m)| \to 0, \quad m \to \infty.$

Riemann–Lebesgue on the Torus

Lemma (Riemann–Lebesgue on $\mathbb{T}$)

For $f \in L^1(\mathbb{T})$, we have $|\mathcal{F}(f)(m)| \to 0, \quad m \to \infty.$

The Fejér kernel is an approximation of the identity on the Torus and a trigonometrical polynomial:

$$\mathfrak{F}_{\mathrm{sp}}(x) = \frac{\sin^2\left(\frac{nx}{2}\right)}{n \sin^2\left(\frac{x}{2}\right)} = \sum_{|k| \leq n-1} \left(1 - \frac{|k|}{n}\right) e^{\mathrm{i}kx}, \quad n \in \mathbb{N}.$$

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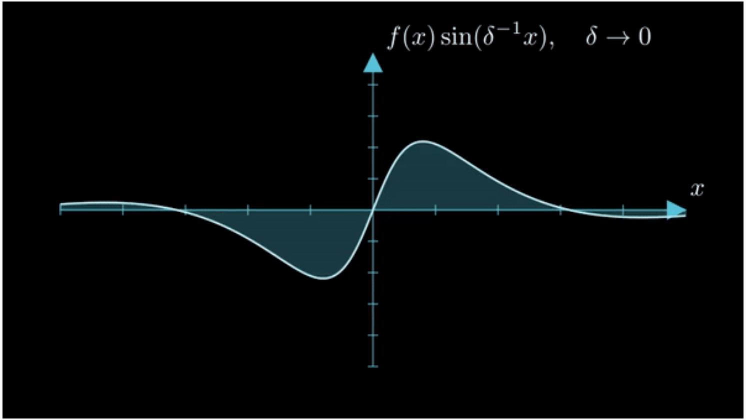
Let P be a trigonometric polynomial with $\|f - P\|_{L^1(\mathbb{T})} < \varepsilon$. If $|m|$ is larger than the degree of the polynomial, then $\mathcal{F}(P)(m) = 0$ and we observe

$$|\mathcal{F}(f)(m)| = |\mathcal{F}(f)(m) - \mathcal{F}(P)(m)| \leq \|f - P\|_{L^1(\mathbb{T})} < \varepsilon.$$

Lemma (Riemann–Lebesgue on $\mathbb{R}^d$)

For $f \in L^1(\mathbb{R}^d)$, we have

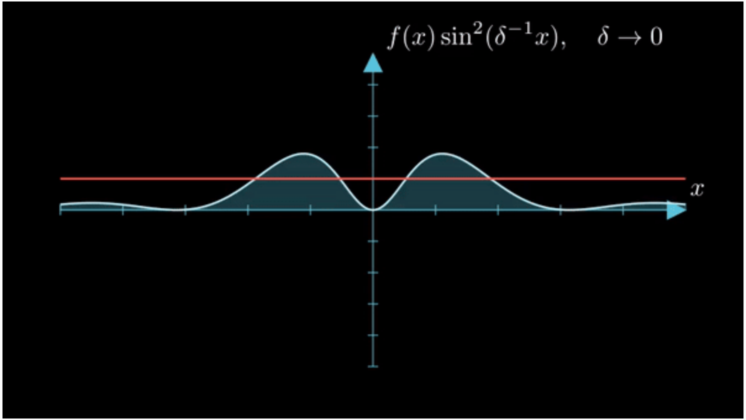
$$\int_{\mathbb{R}^d} \sin(\delta^{-1}x) f(x) \mathrm{d}x \to 0, \quad \delta \to 0.$$



Lemma (Generalised Riemann–Lebesgue Lemma, Kahane [1])

For $f \in L^1(\mathbb{R})$ and $\beta \in L^\infty(\mathbb{R})$ symmetric, we have

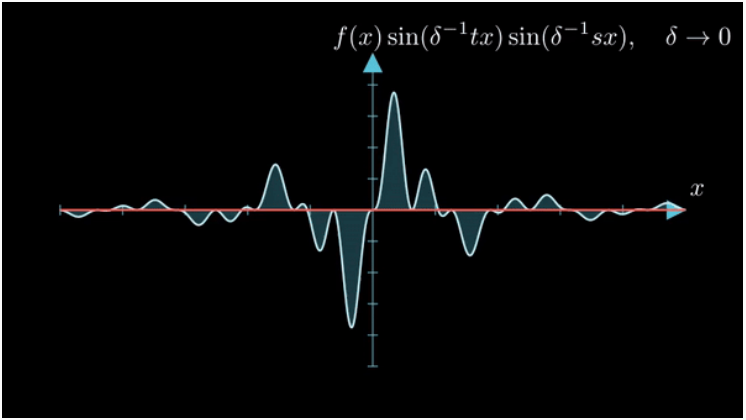
$$\int_{\mathbb{R}} \beta(\delta^{-1}x)f(x)\mathrm{d}x \to 2M(\beta)\int_{\mathbb{R}} f(x)\mathrm{d}x, \quad \delta \to 0, \quad M(\beta) := \lim_{\delta\to 0} \delta \int_0^{\delta^{-1}} \beta(x)\mathrm{d}x.$$



Lemma (Generalised Riemann–Lebesgue Lemma, Kahane [1])

For $f \in L^1(\mathbb{R})$ and $\beta \in L^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \beta(\delta^{-1}x)f(x)\mathrm{d}x \to \left(\int_0^{+\infty} f(t)\mathrm{d}t\right) M(\beta_+) + \left(\int_{-\infty}^{0} f(t)\mathrm{d}t\right) M(\beta_-), \quad \delta \to 0.$$



Riemann–Lebesgue: Deterministic wave equation

The solution to the deterministic wave equation

$$\partial_{tt}^2 u(t,x) = \vartheta \partial_{xx}^2 u(t,x), \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = v_0(x), \quad x \in \mathbb{R}, \quad t \geq 0,$$

is given by d'Alembert's formula

$$u(t,x) = \frac{1}{2}[u_0(x-\sqrt{\vartheta}t) + u_0(x+\sqrt{\vartheta}t)] + \frac{1}{2\sqrt{\vartheta}} \int_{x-\sqrt{\vartheta}t}^{x+\sqrt{\vartheta}t} v_0(\xi)\mathrm{d}\xi$$

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$$\begin{aligned} u(t,x) &= \frac{1}{2}[u_0(x - \sqrt{\vartheta}t) + u_0(x + \sqrt{\vartheta}t)] + \frac{1}{2\sqrt{\vartheta}} \int_{x-\sqrt{\vartheta}t}^{x+\sqrt{\vartheta}t} v_0(\xi)\mathrm{d}\xi \\ &= [C(t)u_0](x) + [S(t)v_0](x), \end{aligned}$$

where $C(t)$ and $S(t)$ are the cosine and sine families associated to the wave equation.

Intuition

$$\cos(x) = (e^{ix} + e^{-ix})/2, \quad C(t) = \cos(t\mathrm{i}\sqrt{\vartheta}\partial_x) = \frac{1}{2}(T(t) + T(-t)).$$

Riemann–Lebesgue: Deterministic wave equation

The total energy $\mathcal{E}$ remains constant in time:

$$\underbrace{\|\sqrt{\vartheta}\partial_x u(t)\|^2_{L^2(\mathbb{R})}}_{\text{potential energy}} + \underbrace{\|\partial_t u(t)\|^2_{L^2(\mathbb{R})}}_{\text{kinetic energy}} \equiv \mathcal{E} := \|\sqrt{\vartheta}\partial_x u_0\|^2_{L^2(\mathbb{R})} + \|v_0\|^2_{L^2(\mathbb{R})}, \quad t \geq 0.$$

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Let $v_0 = 0$. Then,

$$\vartheta\|\partial_x u(t)\|^2_{L^2(\mathbb{R})} - \|\partial_t u(t)\|^2_{L^2(\mathbb{R})} = 4\vartheta \int_{\mathbb{R}} u_0'(\xi) u_0'(\xi + 2\sqrt{\vartheta} t) \mathrm{d}\xi = \frac{2\vartheta}{\pi} \int_{\mathbb{R}} |\hat{u}_0(\omega)|^2 e^{2\mathrm{i}\sqrt{\vartheta} t\omega} \mathrm{d}\omega \to 0$$

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Equipartition of energy

$$\lim_{t\to\infty} \|\sqrt{\vartheta}\partial_x u(t)\|^2_{L^2(\mathbb{R})} = \lim_{t\to\infty} \|\partial_t u(t)\|^2_{L^2(\mathbb{R})} = \frac{\mathcal{E}}{2}.$$

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An operator A such as $\mathrm{i}\sqrt{\vartheta}\partial_x$ satisfying $e^{itA} \to 0$ as $|t| \to \infty$ in the weak operator topology is called a Riemann–Lebesgue operator.

Riemann–Lebesgue: Stochastic wave equation

The stochastic wave equation with unknown parameter $\vartheta > 0$:

$$\partial_{tt}^2 u(t,x) = \vartheta \partial_{xx}^2 u(t,x) + \dot{W}(t,x), \quad x \in \Lambda \subset \mathbb{R}, \quad t \geq 0.$$

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Solution of stochastic wave equation on the unbounded domain $\Lambda = \mathbb{R}$ with Riesz noise $\mathbb{E}[\dot{W}_\beta(t,x)\dot{W}_\beta(s,y)] = |x-y|^{-\beta}\delta(t-s)$ for ($\beta \in (0,1)$; $u_0 = v_0 = 0$):

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The spatial solution process $(u(t,x), x \in \mathbb{R})$ is stationary with the covariance function

$$\zeta(y) = \mathrm{Cov}(u(t,x), u(t,x+y)) = \frac{t}{\pi \vartheta^{\beta/2}} \int_{\mathbb{R}} \cos(y\omega/\sqrt{\vartheta})\,(1 - \mathrm{sinc}(2t\omega))\,|\omega|^{\beta-3}\mathrm{d}\omega.$$

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Spatial ergodicity

$$\mathbb{V}\left(\frac{1}{2L}\int_{-L}^{L} u(t,x)\mathrm{d}x\right) = \frac{t}{4\pi\vartheta^{\beta/2}} \int_{-1}^{1}\int_{-1}^{1} \zeta(L(x-y))\mathrm{d}x\mathrm{d}y \to 0, \quad L \to \infty.$$

Riemann–Lebesgue: Spectral observation scheme

The solution of the stochastic wave equation ($\Lambda = [0, \pi]$) can be represented through harmonic oscillators:

$$u(t,x) = \sum_{k\in\mathbb{N}} u_k(t)e_k(x), \quad u_k(t) = \int_0^t \frac{\sin(\sqrt{\vartheta k}(t-s))}{\sqrt{\vartheta k}} \mathrm{d}w_k(s),$$

where $e_k(x) = \sqrt{2/\pi}\sin(kx)$; w_k are independent Brownian motions and u_k satisfies

$$\mathrm{d}u_k(t) = v_k(t)\mathrm{d}t, \quad \mathrm{d}v_k(t) = -\vartheta k^2 u_k(t)\mathrm{d}t + \mathrm{d}w_k(t).$$

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Maximum likelihood estimator

$$\hat{\vartheta}_N = \frac{-\sum_{k=1}^N \int_0^T k^2 u_k(t) \mathrm{d}v_k(t)}{\sum_{k=1}^N \int_0^T k^4 u_k(t)^2 \mathrm{d}t} = \vartheta - \frac{M_N}{I_N},$$

with

$$M_N := \sum_{k=1}^N k^2 \int_0^T u_k(t) \mathrm{d}w_k(t), \quad I_N := \sum_{k=1}^N \int_0^T k^4 u_k(t)^2 \mathrm{d}t.$$

Riemann–Lebesgue: Spectral observation scheme

The Riemann–Lebesgue lemma implies that

$$\mathbb{E}\left(\int_0^T k^2 u_k^2(t)\mathrm{d}t\right)$$

$$= \frac{1}{\vartheta}\int_0^\infty (T-s)\mathbb{1}_{[0,T]}(s)\sin^2(\sqrt{\vartheta}ks)\mathrm{d}s \to \frac{T^2}{2\vartheta}\left(\lim_{L\to\infty}\frac{1}{L}\int_0^L \sin^2(s)\mathrm{d}s\right) = \frac{T^2}{4\vartheta},$$

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&\mathbb{V}\left(\int_0^T k^2 u_k^2(t)\mathrm{d}t\right)\\
&\quad = \frac{2}{\vartheta^2}\int_0^T\int_0^T\left(\int_0^\infty \mathbb{1}_{[0,t\wedge s]}(r)\sin(\sqrt{\vartheta}k(t-r))\sin(\sqrt{\vartheta}k(s-r))\mathrm{d}r\right)^2\mathrm{d}s\mathrm{d}t \to 0.
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Theorem (Liu and Lototsky [2])

$$N^{3/2}(\hat{\vartheta}_N-\vartheta)\xrightarrow{d}\mathcal{N}(0,12\vartheta T^{-2}),\quad N\to\infty.$$

Riemann–Lebesgue: Local observation scheme

Local measurements

$$u_\delta(t) = \langle u(t,\cdot), K_\delta \rangle_{L^2(\Lambda)}, \quad u_\delta^\Delta(t) = \langle u(t,\cdot), \delta^{-2}(K'')_\delta \rangle_{L^2(\Lambda)}, \quad v_\delta(t) = \langle v(t,\cdot), K_\delta \rangle,$$

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satisfy the dynamics

$$\mathrm{d}u_\delta(t) = v_\delta(t)\mathrm{d}t, \quad \mathrm{d}v_{\delta,x_0}(t) = \vartheta u_\delta^\Delta(t)\mathrm{d}t + \|K\|\mathrm{d}\overline{W}(t),$$

where $(\overline{W}(t), t \in [0, T])$ is a scalar Brownian motion.

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Augmented MLE

$$\hat{\vartheta}_\delta = \frac{\int_0^T u_\delta^\Delta(t)\mathrm{d}v_\delta(t)}{\int_0^T u_\delta^\Delta(t)^2\mathrm{d}t} = \vartheta + \|K\|_{L^2(\mathbb{R})}\frac{M_\delta}{I_\delta}, \quad M_\delta := \int_0^T u_\delta^\Delta(t)\mathrm{d}\overline{W}(t), \quad I_\delta = \int_0^T u_\delta^\Delta(t)^2\mathrm{d}t.$$

Riemann–Lebesgue: Local observation scheme

$$\mathbb{E}[\delta^2 I_\delta] = \int_0^T \int_0^t \|S(\delta^{-1}r)K''\|^2_{L^2(\mathbb{R})} \mathrm{d}r\mathrm{d}t \to \frac{T^2\|K'\|^2_{L^2(\mathbb{R})}}{4\vartheta},$$

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$$\mathbb{V}(\delta^2 I_\delta) = \int_0^T \int_0^T \left(\int_0^{t\wedge s} \langle S(\delta^{-1}(t-r))K'', S(\delta^{-1}(s-r))K'' \rangle_{L^2(\mathbb{R})} \mathrm{d}r \right)^2 \mathrm{d}s\mathrm{d}t \to 0, \quad \delta \to 0.$$

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$$\mathbb{V}(\delta^2 I_\delta) = \int_0^T \int_0^T \left(\int_0^{t\wedge s} \langle S(\delta^{-1}(t-r))K'', S(\delta^{-1}(s-r))K'' \rangle_{L^2(\mathbb{R})} \mathrm{d}r \right)^2 \mathrm{d}s\mathrm{d}t \to 0, \quad \delta \to 0.$$

Plancherel theorem and the Riemann–Lebesgue lemma imply that

$$\delta^2 \mathbb{E}[u_\delta^\Delta(t) u_\delta^\Delta(s)] = \frac{1}{2\pi\vartheta} \int_0^{t\wedge s} \int_{\mathbb{R}} \sin(\delta^{-1}(t-r)\sqrt{\vartheta}|\omega|) \sin(\delta^{-1}(s-r)\sqrt{\vartheta}|\omega|) |\mathcal{F}(K')|^2(\omega) \mathrm{d}\omega \mathrm{d}r \to 0.$$

Riemann–Lebesgue: Local observation scheme

$$\mathbb{E}[\delta^2 I_\delta] = \int_0^T \int_0^t \|S(\delta^{-1}r)K''\|^2_{L^2(\mathbb{R})} \mathrm{d}r\mathrm{d}t \to \frac{T^2\|K'\|^2_{L^2(\mathbb{R})}}{4\vartheta},$$

$$\mathbb{V}(\delta^2 I_\delta) = \int_0^T \int_0^T \left(\int_0^{t\wedge s} \langle S(\delta^{-1}(t-r))K'', S(\delta^{-1}(s-r))K''\rangle_{L^2(\mathbb{R})} \mathrm{d}r\right)^2 \mathrm{d}s\mathrm{d}t \to 0, \quad \delta \to 0.$$

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This can also be seen on the level of operators

$$\|S(\delta^{-1}r)K''\|^2_{L^2(\mathbb{R})} = \frac{1}{\vartheta}\|\sin(\delta^{-1}r\sqrt{\vartheta}\partial_x)K'\|^2_{L^2(\mathbb{R})} = \frac{1}{\vartheta}\langle \sin^2(\delta^{-1}r\sqrt{\vartheta}\partial_x)K', K'\rangle_{L^2(\mathbb{R})} \to \frac{\|K'\|^2_{L^2(\mathbb{R})}}{2\vartheta}.$$

Spatially varying setting

Consider the stochastic wave equation with the spatially varying parameter $\vartheta : \Lambda \to (0, \infty)$:

$$\partial_{tt}^2 u(t,x) = A_\vartheta u(t,x) + \dot{W}(t,x), \quad A_\vartheta z \coloneqq \mathrm{div}(\vartheta \nabla z) \quad t \in [0,T], \quad x \in \Lambda \subset \mathbb{R}^d,$$

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Theorem (Ziebell [5])

Under regularity conditions on ϑ, u_0, v_0 and assumptions on the kernel K, the augmented maximum likelihood estimator

$$\hat{\vartheta}_\delta(x_0) := \frac{\int_0^T u_{\delta,x_0}^\Delta(t)\mathrm{d}v_{\delta,x_0}(t)}{\int_0^T (u_{\delta,x_0}^\Delta(t))^2 \mathrm{d}t}, \quad \delta > 0,$$

satisfies

$$\delta^{-1}(\hat{\vartheta}_\delta(x_0) - \vartheta(x_0)) \xrightarrow{d} \mathcal{N}\left(0, \frac{4\vartheta(x_0)\|K\|_{L^2(\mathbb{R}^d)}^2}{T^2\|\nabla K\|_{L^2(\mathbb{R}^d)}^2}\right).$$

What is the influence of damping?

Consider the stochastic plate equation with unknown parameters $\theta_1, \eta_1 > 0$:

$$\text{(weak) } \mathrm{d}v(t) = (-\theta_1\Delta^2 u(t) + \eta_1 v(t))\mathrm{d}t + \mathrm{d}W(t), \qquad \begin{pmatrix} N^{1/2}\delta^{-2}(\hat\theta_\delta - \theta_1) \\ N^{1/2}(\hat\eta_\delta - \eta_1) \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma_1),$$

$$\text{(structural) } \mathrm{d}v(t) = (-\theta_1\Delta^2 u(t) + \eta_1 \Delta v(t))\mathrm{d}t + \mathrm{d}W(t), \qquad \begin{pmatrix} N^{1/2}\delta^{-1}(\hat\theta_\delta - \theta_1) \\ N^{1/2}\delta^{-1}(\hat\eta_\delta - \eta_1) \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma_2).$$

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Generalises to second-order stochastic Cauchy problems (Tiepner and Ziebell [4]) of the form

$$\partial_{tt}^2 u(t,x) = A_\theta u(t,x) + B_\eta \partial_t u(t,x) + \dot{W}(t,x), \quad t \in [0,T], \quad x \in \Lambda \subset \mathbb{R}^d,$$

with the elasticity and damping operators

$$A_\theta = \sum_{i=1}^{p} \theta_i (-\Delta)^{\alpha_i}, \quad \alpha_1 > \cdots > \alpha_p \geq 0, \quad B_\eta = \sum_{j=1}^{q} \eta_j (-\Delta)^{\beta_j}, \quad \beta_1 > \cdots > \beta_q \geq 0.$$

Riemann–Lebesgue: Discrete observation scheme

$$\partial_{tt}^2 u(t,x) = \vartheta \Delta u(t,x) + \dot{W}_\beta(t,x), \mathbb{E}[\dot{W}_\beta(t,x)\dot{W}_\beta(s,y)] = \delta_0(t-s)|x-y|^{-\beta}, \quad x,y \in \mathbb{R}^d, \quad t,s \geq 0.$$

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Discrete observations

$$\begin{aligned} \text{Spactial: } & (u(t,x_k), k = 0,\dots,n+1), \quad x_k = \lambda k \rho, \quad \rho \in \mathbb{R}^d \setminus \{0\}, \quad |\rho| = 1 \quad \lambda > 0, \\ \text{Temporal: } & (u(t_i,x), i = 0,\dots,m+1), \quad t_i = \delta i, \quad \delta > 0, \\ \text{Spacio-temporal: } & (u(t_i,x_k), i = 0,\dots,m+1, k = 0,\dots,n+1). \end{aligned}$$

Riemann–Lebesgue: Discrete observation scheme

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Second-order variations of the solution process in space and time:

$$\begin{aligned} \mathbf{V}_{\mathrm{sp}} &:= \sum_{k=1}^{n} \mathbf{I}^2_{\mathrm{sp},k}, & \mathbf{I}_{\mathrm{sp},k} &:= u(t,x_{k+1}) + u(t,x_{k-1}) - 2u(t,x_k), \quad k = 1,\dots,n, \\ \mathbf{V}_{\mathrm{te}} &:= \sum_{i=1}^{m} \mathbf{I}^2_{\mathrm{te},i}, & \mathbf{I}_{\mathrm{te},i} &:= u(t_{i+1},x) + u(t_{i-1},x) - 2u(t_i,x), \quad i = 1,\dots,m, \\ \mathbf{V}_{\mathrm{sp,te}} &:= \sum_{i=1}^{m}\sum_{k=1}^{n} \mathbf{I}^2_{\mathrm{sp,te},i,k}, & \mathbf{I}_{\mathrm{sp,te},i,k} &:= \mathbf{I}_{\mathrm{sp},k}(t_{i+1}) + \mathbf{I}_{\mathrm{sp},k}(t_{i-1}) - 2\mathbf{I}_{\mathrm{sp},k}(t_i). \end{aligned}$$

$$\mathbb{E}(n^{-1}m^{-2}\delta^{-1}\lambda^{\beta-2}\mathbf{V}_{\mathrm{sp,te}}) \asymp \frac{1}{\vartheta}\int_0^{\infty} \sin^4(\alpha\sqrt{\vartheta}\omega/2) g_{\mathrm{sp}}(\omega)\mathrm{d}\omega.$$

$$\mathbb{E}(n^{-1}m^{-2}\delta^{-1}\lambda^{\beta-2}\mathbf{V}_{\mathrm{sp,te}}) \asymp \frac{1}{\vartheta}\int_0^\infty \sin^4(\alpha\sqrt{\vartheta}\omega/2)g_{\mathrm{sp}}(\omega)\mathrm{d}\omega.$$

Idea: Use asymptotic properties of the Fejér type

$$\frac{\lambda^{2\beta-4}}{n}\mathbb{V}(\mathbf{V}_{\mathrm{sp}}) \asymp \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathfrak{F}_{\mathrm{sp}}(\omega_1,\omega_2)g_{\mathrm{sp},\lambda}(\omega_1)g_{\mathrm{sp},\lambda}(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

$$\frac{\delta^{2\beta-6}}{m^3}\mathbb{V}(\mathbf{V}_{\mathrm{te}}) \asymp \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathfrak{F}_{\mathrm{te}}(\omega_1,\omega_2)g_{\mathrm{te}}(\omega_1)g_{\mathrm{te}}(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

$$\frac{\delta^{-2}\lambda^{2\beta-4}}{nm^3}\mathbb{V}(\mathbf{V}_{\mathrm{sp,te}}) \asymp \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathfrak{F}_{\mathrm{sp}}(\omega_1,\omega_2)\mathfrak{F}_{\mathrm{te}}(\alpha\sqrt{\vartheta}\omega_1,\alpha\sqrt{\vartheta}\omega_2) \cdot \sin^4(\alpha\sqrt{\vartheta}|\omega_1|/2)g_{\mathrm{sp}}(\omega_1)\sin^4(\alpha\sqrt{\vartheta}|\omega_2|/2)g_{\mathrm{sp}}(\omega_2)\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

with the Fejér type kernels

$$\mathfrak{F}_{\mathrm{sp}}(\omega_1,\omega_2) := \mathfrak{F}_{\mathrm{sp}}(\rho\cdot(\omega_1+\omega_2)) + \mathfrak{F}_{\mathrm{sp}}(\rho\cdot(\omega_1-\omega_2)),$$

$$\mathfrak{F}_{\mathrm{te}}(\omega_1,\omega_2) := \mathfrak{F}_{\mathrm{te}}(|\omega_1|+|\omega_2|) + \mathfrak{F}_{\mathrm{te}}(|\omega_1|-|\omega_2|),$$

$$\mathfrak{F}_{\mathrm{sp}}(x) := \sum_{|k|\le n-1} w_{\mathrm{sp},k}e^{\mathrm{i}kx}, \quad w_{\mathrm{sp,k}} := 1-\frac{|k|}{n}, \quad \mathfrak{F}_{\mathrm{te}}(x) := \sum_{|j|\le m-1} w_{\mathrm{te},j}e^{\mathrm{i}jx}, \quad w_{\mathrm{te},j} := \frac{1}{m^3}\sum_{i=1}^{m-|j|} i^2.$$

Theorem (Tiepner, Trabs, and Ziebell [3])

Under suitable assumptions on the rate of convergence of δ, λ, $\alpha = \delta/\lambda$, m, n:

$$\hat{\vartheta}_{\mathrm{sp},n} := \frac{tC_{\mathrm{sp},\mathbb{E}}}{n^{-1}\lambda^{\beta-2}\mathbf{V}_{\mathrm{sp}}} \qquad \hat{\vartheta}_{\mathrm{te},m} := \left(\frac{C_{\mathrm{te},\mathbb{E}}}{m^{-2}\delta^{\beta-3}\mathbf{V}_{\mathrm{te}}}\right)^{2/\beta},$$

$$\hat{\vartheta}_{\mathrm{box,sp},m,n} := \frac{C_{\mathrm{box,sp},\mathbb{E}}}{n^{-1}m^{-2}\delta^{-1}\lambda^{\beta-2}\mathbf{V}_{\mathrm{sp,te}}}, \quad \hat{\vartheta}_{\mathrm{box,te},m,n} := \left(\frac{C_{\mathrm{box,te},\mathbb{E}}}{n^{-1}m^{-2}\delta^{\beta-3}\mathbf{V}_{\mathrm{sp,te}}}\right)^{2/\beta},$$

satisfy the central limit theorems

$$\sqrt{n}(\hat{\vartheta}_{\mathrm{sp},n} - \vartheta) \xrightarrow{d} \mathcal{N}\left(0, \vartheta^2 \frac{C_{\mathrm{sp},\mathbb{V}}}{(C_{\mathrm{sp},\mathbb{E}})^2}\right), \quad n \to \infty,$$

$$\sqrt{m}(\hat{\vartheta}_{\mathrm{te},m} - \vartheta) \xrightarrow{d} \mathcal{N}\left(0, \vartheta^2 \frac{4C_{\mathrm{te},\mathbb{V}}}{\beta^2 C_{\mathrm{te},\mathbb{E}}^2}\right), \quad m \to \infty,$$

$$\sqrt{nm}(\hat{\vartheta}_{\mathrm{box,sp},m,n} - \vartheta) \xrightarrow{d} \mathcal{N}\left(0, \vartheta^2 \frac{C_{\mathrm{box,sp},\mathbb{V}}}{(C_{\mathrm{box,sp},\mathbb{E}})^2}\right), \quad \alpha \to \infty,$$

$$\sqrt{nm}(\hat{\vartheta}_{\mathrm{box,te},m,n} - \vartheta) \xrightarrow{d} \mathcal{N}\left(0, \frac{4C_{\mathrm{box,te},\mathbb{E}}}{\beta^2 C_{\mathrm{box,te},\mathbb{E}}^2}\right), \quad \alpha \to 0.$$

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